Spring 2017 MATH5012

Real Analysis II

Exercise 6

(1) Let $f \in L^1(\mu)$ on some measure space (X, μ). Show that for each $\varepsilon > 0$, there exists some δ such that

$$\int_E |f| < \varepsilon, \quad \text{for all } \mu\text{-measurable } E, \mu(E) < \delta \ .$$

(2) Setting as in (1), a class of integrable functions $\mathcal{C}$ is called uniformly integrable if the δ in (1) can be chosen to fit for all f in $\mathcal{C}$. Show that $\mathcal{C}$ is uniformly integrable if and only if, for every $\varepsilon > 0$, there is some M such that

$$\int_{\{x:|f|\geq M\}} |f| d\mu < \varepsilon, \quad \forall f \in \mathcal{C} \ .$$

(3) Let $\{f_k\}$ be a sequence of integrable functions in (X, μ) satisfying

$$\int |f_k| \log(1 + |f_k|) d\mu \leq M \ , \quad \forall k \geq 1 \ ,$$

for some M. Show that $\{f_k\}$ is uniformly integrable.

(4) (a) Display a sequence of integrable functions in $[-1, 1]$ whose L^1-norm is uniformly bounded and yet does not subconverge to any integrable function.

(b) Display a sequence of uniformly integrable functions in $\mathbb{R}$ which does not subconverge to any integrable function.